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# A Study of Positive Linear Operators by the Method of Moments, One-Dimensional Case*

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Let $[a, b] \subset R$ and let $\{L_j\}_{j \in \mathbb{N}}$ be a sequence of positive linear operators from $C^n[a, b]$ $(n \in \mathbb{Z}^+)$ to $C[a, b]$. The convergence of $L_j$ to the identity operator $I$ is closely related to the weak convergence of a sequence of finite measure $\mu_j$ to the unit (Dirac) measure $\delta_{x_0}$, $x_0 \in [a, b]$. New estimates are given for the remainder $|\int_{[a,b]} f\, d\mu_j - f(x_0)|$, where $f \in C^n([a, b])$. Using moment methods, Shisha–Mond-type best or nearly best upper bounds are established for various choices of $[a, b]$, $n$ and given moments of $\mu_j$. Some of them lead to attainable inequalities. The optimal functions/measures are typically spline functions and finitely supported measures. The corresponding inequalities involve the first modulus of continuity of $f^{(n)}$ (the $n$th derivative of $f$) or a modification of it. Several applications of these results are given. 

## PART 1. INTRODUCTION

We start with the following definition:

DEFINITION 1.1. Let $Q$ be a connected compact Hausdorff space and $C(Q, \mathbb{R})$ the collection of all continuous $f: Q \to \mathbb{R}$. Let $g \in C(Q, \mathbb{R})$ be fixed and define the $g$-pseudomodulus of continuity of $f \in C(Q, \mathbb{R})$ as

$$w_g(f, h) = \sup_{x, y}\{|f(x) - f(y)|: |g(x) - g(y)| \leqslant h\}, \tag{1.1.1}$$

where $h \geqslant 0$.

Thus $w_g(g, h) \leqslant h$. The quantity $w_g(f, h)$ enjoys most of the basic properties of the usual modulus of continuity $\omega_1(f, h)$ (positively homogeneous as

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a function of $f$, non-decreasing, non-negative, and subadditive in $h$). However, $w_g(f, \cdot)$ is an upper-semicontinuous function and in general not a continuous one.

EXAMPLE. Let

$$g(x)=0, \qquad 0 \leqslant x \leqslant 1;$$
$$= x-1, \qquad 1 \leqslant x \leqslant 2;$$
$$= 1, \qquad 2 \leqslant x \leqslant 3$$

and $f(x)=x$ then

$$w_g(f, h)=1+h, \qquad 0 \leqslant h<1;$$
$$=3, \qquad h \geqslant 1.$$

Obviously, $w_g(f, \cdot)$ is discontinuous.

Consider a sequence of positive linear operators $L_n$: $C(Q, \mathbb{R}) \rightarrow C(Q, \mathbb{R})$, such that the sequence of functions $\{L_n(1)\}_{n \in \mathbb{N}}$ is uniformly bounded. In particular, $|f| \leqslant \tilde{g}$ implies $|L_n(f)| \leqslant L_n(\tilde{g})$. The following result is an easy generalization of a result due to Shisha and Mond [26], who took $Q=[a, b] \subset \mathbb{R}$ and $g(x)=x$.

THEOREM 1.2. *One has*

$$\|L_n(f)-f\| \leqslant\|f\|\,\|L_n(1)-1\|+w_g(f, \rho_n)(1+\|L_n(1)\|), \tag{1.2.1}$$

*where*

$$\rho_n=(\|L_n((g-g(y))^2)(y)\|)^{1/2}.$$

*Further,* $\|\cdot\|$ *stands for the supremum norm. If* $L_n(1)=1$, *then* (1.2.1) *simplifies to*

$$\|L_n(f)-f\| \leqslant 2 w_g(f, \rho_n).$$

*As an application, one has the following well-known theorem due to Korovkin* [18].

COROLLARY 1.3. *Let* $Q=[a, b] \subset \mathbb{R}$ *and let* $\{L_n: C([a, b]) \rightarrow C([a, b])\}_{n \in \mathbb{N}}$ *be a sequence of positive linear operators. Suppose that* $g \in C([a, b])$ *is* 1:1 *and further that* $L_n(1) \rightarrow^u 1$, $L_n(g) \rightarrow^u g$, *and* $L_n(g^2) \rightarrow^u g^2$. *Then* $L_n(f) \rightarrow^u f$, *for all* $f \in C([a, b])$.

*Proof.* Note that

$$\rho_n^2 \leqslant \|L_n(g^2) - g^2\| + 2\,\|g\|\,\|L_n(g) - g\| + \|g\|^2\,\|L_n(1) - 1\|.$$

Now apply Theorem 1.2. ∎

Z. Ditzian [8] gave extensions of (1.2.1) to the non-compact case under suitable growth conditions for $f$.

Also R. A. DeVore [7] gave analogues of (1.2.1) for $f \in C^1([a, b])$. Furthermore, E. Censor [6] extended (1.2.1) to the multidimensional case and established related results for $f \in C^2([a, b])$. Next B. Mond [22] gave a more flexible inequality which sometimes leads to better constants in the upper bounds for $\|L_n(f) - f\|$. This result was carried over to $f \in C^1([a, b])$, by B. Mond and R. Vasudevan [23].

The latest result in this direction which often gives better constants as well as a higher degree of approximation for $f \in C^1([a, b])$ is due to H. H. Gonska [12]. He used the first Stekloff function of $f'$.

J. P. King [17] gave a probabilistic interpretation of Korovkin's main theorem and certain pointwise inequalities analogous to (1.2.1) for $f \in C([a, b])$ or $f \in C^1([a, b])$.

We have mentioned only the papers directly related to our research. However, there is a large related literature, for instance, the significant theoretical work in Korovkin theory by J. A. Šaškin [24], G. G. Lorentz [21], D. Amir and Z. Ziegler [1], H. Bauer [4], and most recently K. Donner [9].

Our method is to reduce questions about positive linear operators to questions about finite (positive) measures. Namely, let $L: C[a, b] \to C[a, b]$ be a positive linear operator. Then for any $x \in [a, b]$ there is a finite measure $\mu_x$ such that

$$L(f, x) = \int f(t)\,\mu_x(dt), \qquad \text{for all } f \in C[a, b].$$

And many questions can be reduced to moment problems involving the measure $\mu_x$. Using standard moment methods, see [15, 16], we derive pointwise estimates for $|L(f, x) - f(x)|$ which sometimes imply uniform ones. The advantage of this approach is that frequently one even obtains attainable (i.e., sharp) or nearly attainable inequalities. The optimal elements $f$, $\mu_x$ are often spline functions and finitely supported measures, respectively. Thus, this paper mainly deals with the quantitative study of the pointwise convergence of a sequence of positive linear operators to the identity operator through the use of moment methods.

## PART 2. ONE-DIMENSIONAL QUANTITATIVE RESULTS FOR FINITE APPROXIMATION MEASURES OF THE UNIT MEASURE

In the rest of this paper we study the degree of weak convergence of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}$ to the unit measure $\delta_{x_0}$. In fact we estimate $|\int_Q f\,d\mu - f(x_0)|$; $f\in C^n(Q)$, $n\in\mathbb{Z}^+$, $x_0\in Q$. Here $Q$ is usually a compact interval of $\mathbb{R}$, sometimes $\mathbb{R}$ itself. This enables us in turn to estimate $|L(f,x)-f(x)|$, where $L$ is a positive linear operator $C^n(Q)\mapsto C(Q)$.

Using standard moment methods, we obtain best or nearly best upper bounds, often attainable, for different $Q$, $n$ and given power moments of $\mu$.

Our inequalities involve the first modulus of continuity $\omega_1(f^{(n)},h)$, or a modified version of it, of the $n$th derivative $f^{(n)}$ for a fixed value of the argument $h$.

We present several favorable comparisons of our results to related known results, for instance, the inequality due to O. Shisha and B. Mond [26], as well as to the latest improvement due to H. H. Gonska [12].

### I. Preliminaries

The following general result leads to Corollary 2.2 which is used a lot throughout this paper.

THEOREM 2.1. *Let $C$ be a subset of the real normed vector space $V=(V,\|\cdot\|)$ which is star-shaped relative to the fixed point $x_0$. Let further $\{(h_i,w_i): i\in I\}$ be a given collection of numbers ($h_i>0$, $w_i>0$, $I$ arbitrary), and consider the collection $\mathscr{F}$ of functions $f: C\to\mathbb{R}$ such that $f(x_0)=0$ while, for each $i\in I$,*

$$\|s-t\|\leqslant h_i \Rightarrow |f(s)-f(t)|\leqslant w_i. \tag{2.1.1}$$

*Then*

$$\sup_{f\in\mathscr{F}} |f(s)| = \rho(\|s-x_0\|) \qquad (s\in C), \tag{2.1.2}$$

*where*

$$\rho(\|u\|)=\inf\left\{\sum_{i\in I} k_i w_i \colon \|u\|\leqslant \sum_{i\in I} k_i h_i\right\}$$

*where $k_i\in\mathbb{Z}^+$, $k=\sum_{i\in I}k_i<\infty$.*

*Proof.* Obviously, $\rho(\|u\|)$ is an even subadditive function on $\mathbb{R}$ satisfying $\rho(0)=0$ and $\rho(h_i)\leqslant w_i$ ($i\in I$). Moreover, $\rho(\|u\|)$ is non-decreasing on

$\mathbb{R}^+$. Hence $f_0(s) = \rho(\|s - x_0\|)$ restricted to $C$ defines a function $f_0 \in \mathscr{F}$, showing that (2.1.2) holds with the $\geqslant$ sign. To prove the opposite inequality, it suffices to show that $|f(s)| \leqslant \sum_{i \in I} k_i w_i$ as soon as $f \in \mathscr{F}$ and $\|s - x_0\| \leqslant \sum_{i \in I} k_i h_i$ ($k_i \in \mathbb{Z}^+$, $k = \sum_{i \in I} k_i < \infty$). This is easily done by an induction on $k$. The cases $k = 0$ and $k = 1$ are obvious. Let $k \geqslant 0$ satisfy the assertions and suppose $s \in C$ satisfies

$$\|s - x_0\| \leqslant \sum_{i \in I} k_i h_i + h_r \qquad \left(k_i \in \mathbb{Z}^+, \sum k_i = k; r \in I\right).$$

Choosing $s'$ on the line segment $x_0 s$ such that $\|s' - s_0\| \leqslant \sum_1^n k_i h_i$ and $\|s - s'\| \leqslant h_r$, one easily sees that $|f(s)| \leqslant \sum_1^n k_i w_i + w_r$. ▮

COROLLARY 2.2. *Let $C$ and $x_0$ be as above and consider $f: C \to \mathbb{R}$ with the properties*

$$f(x_0) = 0 \tag{2.2.1}$$

*and*

$$\|s - t\| \leqslant h \Rightarrow |f(s) - f(t)| \leqslant w; \qquad w, h > 0.$$

*Then there is a maximal such function $\phi$, namely,*

$$\phi(t) = \lceil \|t - x_0\|/h \rceil w, \tag{2.2.2}$$

*where $\lceil \cdot \rceil$ indicates the ceiling of the number.*

## 2.3. *An Auxiliary Function*

Let $h > 0$ be fixed. We shall often use the even function defined by

$$\phi_n(x) = \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} \, dt \qquad (x \in \mathbb{R}). \tag{2.3.1}$$

Equivalently,

$$\phi_n(x) = \int_0^{|x|} \int_0^{x_1} \cdots \left( \int_0^{x_{n-1}} \left\lceil \frac{x_n}{h} \right\rceil dx_n \right) \cdots dx_1. \tag{2.3.2}$$

Since $\lceil t/h \rceil = \sum_{j=0}^{\infty} 1_{jh < t}$, the latter yields that

$$\phi_n(x) = \frac{1}{n!} \left( \sum_{j=0}^{\infty} (|x| - jh)_+^n \right). \tag{2.3.3}$$

In particular, letting $k = \lceil |x|/h \rceil$,

$$\phi_1(x) = \sum_{j=0}^{\infty} (|x| - jh)_+ = \sum_{j=0}^{k-1} (|x| - jh) = k\,|x| - \frac{1}{2} k(k-1)\,h. \tag{2.3.4}$$

Maximizing over $k \geqslant 0$, (attained if $k = \frac{1}{2} + |x|/h$), it follows that

$$\phi_1(x) \leqslant \phi_{*1}(x) = \frac{1}{2h}\,(|x| + h/2)^2. \tag{2.3.5}$$

In fact, $\phi_1(x_k) = \phi_{*1}(x_k) = \frac{1}{2} \cdot hk^2$ and $\phi_1'(x_k) = \phi_{*1}'(x_k) = k$ at $x_k = (k - \frac{1}{2})\,h$ $(k = 1, 2, 3, \ldots)$. Further, one easily gets

$$\phi_n(x) \leqslant \phi_{*n}(x) = \left(\frac{|x|^{n+1}}{(n+1)!\,h} + \frac{|x|^n}{2n!} + \frac{h\,|x|^{n-1}}{8(n-1)!}\right) \tag{2.3.6}$$

with equality only at $x = 0$. Note from (2.3.3) that $\phi_n$ is a (polynomial) *spline* function. In each interval $((j-1)\,h, jh]$ equals a polynomial of degree $n$. At the points $jh$ $(j = 0, 1, \ldots)$ the derivatives $D^k \varphi_n$ $(k = 0, 1, \ldots, n-1)$ are continuous while the $n$th derivative makes an upward jump of size 1. Moreover, $\phi_n(x)$ is convex on $\mathbb{R}$ and strictly increasing on $\mathbb{R}^+$ $(n \geqslant 1)$. Finally,

$$\phi_n(x) = \int_0^x \phi_{n-1}(t)\,dt \qquad (x \in \mathbb{R}^+,\ n \geqslant 1) \tag{2.3.7}$$

provided we define $\phi_0(t) = \lceil t/h \rceil$.

## II. Best Upper Bounds and Related Results

Using moment theory methods we obtain the following results.

THEOREM 2.4. *Let $\mu$ be a finite measure of mass $m$ on the interval $[a, b]$ where $0 \in [a, b]$. Let $c = \max(|a|, b)$. Suppose further that*

$$\left(\int |t|^r\,\mu(dt)\right)^{1/r} = d, \tag{2.4.1}$$

*where $r > 0$ and $d > 0$ are given. In order that $\mu$ exists, we also assume that $d^r \leqslant mc^r$. Next, consider $f\colon [a, b] \to \mathbb{R}$ satisfying*

$$|f(s) - f(t)| \leqslant w \qquad \textit{when} \quad s, t \in [a, b];\ |s - t| \leqslant h. \tag{2.4.2}$$

*Here, $h>0$ and $w>0$ are fixed. Then the best possible constant $K=K(m, r, d, h, w, f(0))$ in the inequality*

$$\left|\int f\,d\mu - f(0)\right| \leqslant |m-1|\,|f(0)| + mK \tag{2.4.3}$$

*is given as follows (and is independent of $f(0)$). Here $n=\lceil c/h\rceil$, $k=\lceil d/hm^{1/r}\rceil$. Since $d^r \leqslant mc^r$ one has $1 \leqslant k \leqslant n$.*

(i) *$K=nw$ when $k=n$, that is, when $d/m^{1/r}>c-h$.*

(ii) *$K=[1+(1/m)(d/h)^r (n-1)^{1-r}]\,w$ when $r\leqslant 1$ and $k<n$.*

(iii) *$K=(k+\Theta_k)\,w\leqslant[1+d/(hm^{1/r})]\,w$ when $r\geqslant 1$ and $k<n$. Here*

$$\Theta_k=[d^r/m-(k-1)^r h^r]/[k^r h^r-(k-1)^r h^r].$$

*The equality sign in (2.4.3) is usually not attained but can be approached arbitrarily closely by the function $f(t)=f(0)+\varepsilon\lceil t/h\rceil w$ (with $\varepsilon=\pm 1$ of the same sign as $(m-1)f(0)$) and a measure $\mu$ of mass $m$ supported by a single point in case (i), by at most two points 0 and $t^*$ in case (ii) (with absolute value slightly to the right of $(n-1)h$), and by at most two points $t_1$ and $t_2$ in case (iii) (with absolute value slightly to the right of $(k-1)h$ and $kh$, respectively).*

*Proof.* Let $g(t)=f(t)-f(0)$. From Corollary 2.2, we have

$$|g(t)|\leqslant\phi(t)=\lceil |t|/h\rceil\,w.$$

Thus

$$\left|\int f\,d\mu-f(0)\right|=\left|\int g\,d\mu+(m-1)f(0)\right|\leqslant\int\phi\,d\mu+|m-1|\,|f(0)|.$$

Here, the equality sign obtains when $f$ is of the form $f_0(t)=f(0)+\varepsilon\phi(t)$ with $\varepsilon=\pm 1$ of the same sign as $(m-1)f(0)$. One easily verifies that $f_0$ also satisfies (2.4.2). Thus, the best constant $K$ in (2.4.3) is given by

$$mK=\sup_{\mu}\int\phi\,d\mu$$

where $\mu$ ranges over the measures on $[a, b]$ of mass $m$ which satisfy (2.4.1). Introducing the probability measure $dv=m^{-1}\,d\mu$, we have

$$K=\sup_{v}\int\phi(t)\,v(dt)$$

where $\nu$ ranges through the probability measures on $[a, b]$ satisfying

$$\int |t|^r \nu(dt) = d^r/m.$$

Note that both $\phi(t)$ and $|t|^r$ are even functions on $[a, b]$. It follows (see [15, 16]) that $K = \psi(d^r/m)$, where $\Gamma_1 = \{(u, \psi(u)) \colon 0 \leqslant u \leqslant c^r\}$ describes the upper boundary of the convex hull conv $\Gamma_0$ of the curve

$$\Gamma_0 = \{(t^r, \phi(t)) \colon 0 \leqslant t \leqslant c\} \qquad (\text{where } c = \max(|a|, b)).$$

Note that $\Gamma_0$ consists of the following parts:

(i) The origin $(0, 0)$ corresponding to $t = 0$.

(ii) For $k = 1, \dots, n-1$, the half open horizontal line segments $(P_k, Q_k]$ where $P_k = ((k-1)^r h^r, kw)$ and $Q_k = (k^r h^r, kw)$.

(iii) The half open non-empty horizontal line segment $(P_n, Q_*]$, where $Q_* = (c^r, nw)$. It is always a part of the upper boundary $\Gamma_1$ of conv $\Gamma_0$, yielding assertion (i) of the theorem.

The line segment from $P_k$ to $P_{k+1}$ has a slope $wh^{-r}/[k^r - (k-1)^r]$ which is *decreasing* in $k$ when $r \geqslant 1$ and *increasing* in $k$ when $0 < r \leqslant 1$ (since the latter denominator has derivative $r[k^{r-1} - (k-1)^{r-1}]$). Consequently, if $0 < r \leqslant 1$ then $\Gamma_1$ consists of the two line segments $[P_1, P_n]$ and $[P_n, Q_*]$. Thus $\psi(u) = w[1 + uh^{-r}(n-1)^{-r+1}]$ when $0 \leqslant u \leqslant (n-1)^r h^r$ while $\psi(u) = nw$ when $(n-1)^r h^r \leqslant u \leqslant c^r$. This yields assertion (ii). On the other hand, if $r \geqslant 1$ then $\Gamma_1$ is composed of the line segments $[P_k, P_{k+1}]$ $(k = 1, \dots, n-1)$ together with the horizontal line segment $[P_n, Q_*]$. This easily yields assertion (iii). The last assertions of the theorem easily follow from the geometry of conv $\Gamma_0$. The fact that this sharp bound (2.4.3) is usually not attained derives from the fact that $[P_k, P_{k+1}]$ belongs to the closure of conv $\Gamma_0$ but not to conv $\Gamma_0$ itself. ∎

COROLLARY 2.5. *If* $r \geqslant 1$ *then*

$$\left| \int f \, d\mu - f(0) \right| \leqslant |m - 1| \, |f(0)| + w \left( m + \frac{d}{h} m^{1-(1/r)} \right). \tag{2.5.1}$$

*Note.* Clearly, when $r = 2$ inequality (2.4.3) leads to a sharper inequality than the corresponding Shisha–Mond-type inequality (1.2.1). By a similar reasoning, one obtains the following result.

PROPOSITION 2.6. *Let* $[a, b]$ *be a closed finite interval containing* 0 *and put* $c = \max(|a|, b)$ *and* $L = b - a$. *Let* $\Phi \colon [-L, +L] \to \mathbb{R}^+$ *be an even non-*

*negative function,* $\Phi(0)=0$, *which is subadditive on* $[0, L]$ *(e.g.,* $\Phi$ *could be nondecreasing and concave). Consider*

$$\mathscr{F}=\{f\colon [a,b]\to\mathbb{R} \text{ and } |f(s)-f(t)|\leqslant\Phi(s-t) \text{ for } s,t\in[a,b]\}.$$

*Let* $\mathscr{F}_0=\{f\in\mathscr{F}: f(0)=0\}$. *Clearly* $\Phi\in\mathscr{F}_0$ *and* $|f|\leqslant\Phi$ *for all* $f\in\mathscr{F}_0$ *thus*

$$\sup_{\mathscr{F}_0}\left|\int f\,d\nu\right|=\int\Phi\,d\nu,$$

*for each probability measure* $\nu$ *on* $[a,b]$.

*Suppose* $\nu$ *is further restricted by the moment condition*

$$\int\Psi(t)\,\nu(dt)=\rho$$

*where* $\Psi$ *is a given continuous even and non-negative function on* $[-c,+c]$, *which is strictly increasing on* $[0,c]$, $\Psi(0)=0$. *Let*

$$E=\sup_{\mathscr{F},\nu}\left|\int f\,d\nu\right|.$$

(i) *If* $\Phi(\Psi^{-1})$ *is convex on* $[0,\Psi(c)]$ *then* $E=\rho\Phi(c)/\Psi(c)$.

(ii) *If* $\Phi(\Psi^{-1})$ *is concave on* $[0,\Psi(c)]$ *then* $E=\Phi(\Psi^{-1}(\rho))$.

*Remark* 2.7. If $\Phi(t)=k\,|t|^{\alpha}$ $(0<\alpha\leqslant 1, k>0)$, $\Psi(t)=|t|^{r}$ $(r>0)$ then (i), (ii) happen when $r\leqslant\alpha$ and $r\geqslant\alpha$, respectively.

*Remark* 2.8. We maintain the notations of Proposition 2.6. Let $\mu$ be a measure on $[a,b]$ with mass $m$ and moment

$$\int\Psi(t)\,\mu(dt)=m\rho.$$

Then each $f\in\mathscr{F}$ satisfies

$$\left|\int f\,d\mu-f(0)\right|\leqslant|f(0)|\,|m-1|+mE.$$

This inequality is sharp and in fact is attained by $f(t)=f(0)+\varepsilon\Phi(t)$ and a suitably chosen measure supported by at most two points ($\varepsilon=\pm1$ equals the sign of $(m-1)\,f(0)$).

The following is well known.

LEMMA 2.9. *Be given a closed finite interval* $[a,b]$ *and fixed* $x_0\in[a,b]$.

*Consider* $f \in C^n([a, b])$, $n \geqslant 1$, *and denote* $\phi(x) = f^{(n)}(x) - f^{(n)}(x_0)$. *Then* $(a \leqslant x \leqslant b)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \int_{x_0}^{x} \phi(t) \cdot \frac{(x-t)^{n-1}}{(n-1)!} \cdot dt. \tag{2.9.1}$$

As a related result we give

THEOREM 2.10. *Let* $\mu$ *be a measure of mass* $m > 0$ *on* $[a, b]$ *and* $x_0 \in [a, b]$ *fixed. Let n be a fixed positive integer and put*

$$h = \left[ \int |t - x_0|^{n+1} \mu(dt) \right]^{1/(n+1)}. \tag{2.10.1}$$

*Suppose* $f \in C^n([a, b])$ *satisfies*

$$|f^{(n)}(s) - f^{(n)}(t)| \leqslant w \qquad \text{if} \quad a \leqslant s,\ t \leqslant b, \text{ and } |s - t| \leqslant h, \tag{2.10.2}$$

*where* $w$ *is a given positive number. Then*

$$\left| \int f \, d\mu - f(x_0) \right| \leqslant |f(x_0)| \, |m - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}(x_0)|}{k!} \left| \int (t - x_0)^k \mu(dt) \right|$$
$$+ \frac{wh^n}{n!} (m^{1/(n+1)} + 1/(n+1)). \tag{2.10.3}$$

*Proof.* Without loss of generality let $x_0 = 0$. From (2.10.2) $\phi(x) = f^{(n)}(x) - f^{(n)}(0)$ satisfies $|\phi(s) - \phi(t)| \leqslant w$ when $|s - t| \leqslant h$, therefore $|\phi(t)| \leqslant w \lceil |t|/h \rceil$ by Corollary 2.2. It follows from (2.9.1) and (2.3.1) that

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \right| \leqslant w \phi_n(x).$$

From $\lceil t/h \rceil \leqslant 1 + t/h$ and (2.3.1),

$$w\phi_n(x) \leqslant \frac{w |x|^n}{n!} \left( 1 + \frac{|x|}{(n+1) h} \right). \tag{2.10.4}$$

Integrating relative to $\mu$ and using Hölder's inequality we obtain (2.10.3). ∎

PROPOSITION 2.11. *Let* $f \in C^n([-\pi, \pi])$, $n \geqslant 1$, *and* $\mu$ *a measure on* $[-\pi, \pi]$ *of mass* $m > 0$. *Put*

$$\beta = \left( \int \left( \sin \frac{|t|}{2} \right)^{n+1} \mu(dt) \right)^{1/(n+1)} \tag{2.11.1}$$

*and denote by* $w = \omega_1(f^{(n)}, \beta)$ *the modulus of continuity of* $f^{(n)}$ *at* $\beta$. *Then*

$$\left|\int f\,d\mu - f(0)\right| \leqslant |f(0)|\,|m-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!}\left|\int t^k \mu(dt)\right|$$
$$+ w[m^{1/(n+1)} + \pi/(n+1)]\,\frac{\pi^n\beta^n}{n!}. \qquad (2.11.2)$$

*Proof.* Analogous to the proof of Theorem 2.10, using the fact $|t| \leqslant \pi \sin(|t|/2)$. ▮

## III. Attainable Inequalities

The following optimal results are obtained by using standard moment methods (see [15, 16]).

Theorem 2.12. *Let* $\mu$ *be a finite measure on* $[a, b] \subset \mathbb{R}$, $0 \in (a, b)$ *and* $|a| \leqslant b$. *Put*

$$c_k = \int t^k \mu(dt), \qquad k = 0, 1, \ldots, n; \qquad d_n = \left(\int |t|^n\,\mu(dt)\right)^{1/n}. \quad (2.12.1)$$

*Let* $f \in C^n([a, b])$ *be such that*

$$|f^{(n)}(s) - f^{(n)}(t)| \leqslant w \qquad \text{if} \quad a \leqslant s, t \leqslant b, \text{ and } |s-t| \leqslant h \quad (2.12.2)$$

*where* $w$, $h$ *are given positive numbers.*

*Then we have the upper bound*

$$\left|\int f\,d\mu - f(0)\right| \leqslant |f(0)|\,|c_0 - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}_{(0)}|}{k!}\,|c_k| + w\phi_n(b)\left(\frac{d_n}{b}\right)^n. \quad (2.12.3)$$

*The above inequality is in a certain sense attained by the measure* $\mu$ *with masses* $[c_0 - (d_n/b)^n]$ *and* $(d_n/b)^n$ *at* 0 *and* $b$, *respectively, and when, moreover, the optimal function is*

$$\tilde{f} = w\phi_n, \qquad \text{on} \quad [0, b];$$
$$= 0, \qquad \text{on} \quad [a, 0]. \qquad (2.12.4)$$

*Namely, the latter is the limit of a sequence of functions* $f$ *having continuous* $n$*th derivatives satisfying* (2.12.2) *and* $f^{(k)}(0) = 0$ $(k = 0, \ldots, n)$ *and* *such that the difference of the two sides of* (2.12.3) *tends to* 0. *In fact,* $\lim_{N\to+\infty} f_{nN}(t)=\bar{f}(t)$, *where* $(a\leqslant t\leqslant b)$

$$f_{nN}(t)=w\int_0^t\left(\int_0^{t_1}\cdots\left(\int_0^{t_{n-1}} f_{0N}(t_n)\,dt_n\right)\cdots\right)dt_1. \tag{2.12.5}$$

*Here, for* $k=0,1,\ldots,\lceil b/h\rceil-1$ *and* $N\geqslant 1$ $f_{0N}$ *is the continuous function defined*

$$\begin{aligned} f_{0N}(t)&=0, && \text{if } a\leqslant t<0;\\ &=\frac{Nwt}{2h}+kw\left(1-\frac{N}{2}\right), && \text{if } kh\leqslant t\leqslant\left(k+\frac{2}{N}\right)h;\\ &=(k+1)\,w, && \text{if } \left(k+\frac{2}{N}\right)h<t\leqslant(k+1)\,h;\\ &=\lceil b/h\rceil\,w, && \text{if } \left(\lceil b/h\rceil-1+\frac{2}{N}\right)h<t\leqslant b. \end{aligned} \tag{2.12.6}$$

*Observe that* $f_{0N}(t)$ *fulfills* (2.12.2) *and further*

$$\begin{aligned} \lim_{N\to+\infty} f_{0N}(t)&=\lceil t/h\rceil\,w, && t\in[0,b];\\ &=0, && t\in[a,0]. \end{aligned}$$

*Proof.* From (2.9.1), integrating relative to $\mu$ get

$$\left|\int f\,d\mu-f(0)\right|\leqslant|f(0)|\,|c_0-1|+\sum_{k=1}^{n}\frac{|f^{(k)}(0)|}{k!}\,|c_k|+S_n$$

where

$$S_n=w\int\phi_n(t)\,\mu(dt).$$

We would like to maximize $S_n$ given that $\mu$ has preassigned moments $c_0$ and $d_n=[\int|t|^n\,\mu(dt)]^{1/n}$. Since the functions on hand $|t|^n$ and $\phi_n(t)$ are both even we are essentially concerned with a measure on $[0,b]$ (using that $|a|\leqslant b$). As usual, consider the curve defined by $u=t^n$ and $v=\phi_n(t)$, that is, $v=\phi_n(u^{1/n})$ where $u\geqslant 0$. Here

$$\phi_n(u^{1/n})=\frac{1}{n!}\left(\sum_{j=0}^{\infty}(u^{1/n}-jh)_+^n\right).$$

The function $(u^{1/n}-jh)^n_+$ has its first derivative equal to $(1-jh/u^{1/n})^{n-1}_+$ which is obviously increasing in $u$. It follows that $\phi_n(u^{1/n})$ is convex. Consequently, the integral $\int \phi_n(t)\,\mu(dt)$ is maximized by a measure taking values at 0 and $b$ only. Let $\mu$ have masses $p$ and $q$ at 0 and $b$, respectively. Thus $p \geqslant 0$, $q \geqslant 0$ while $p+q=c_0$. Further, $0+qb^n=d_n^n$, thus $q=(d_n/b)^n$. Consequently, $\max S_n = w\phi_n(b)\,q = w\phi_n(b)(d_n/b)^n$. ∎

Let $L$ be a positive linear operator from $C^n([a,b])$ into $C([a,b])$. It follows from the Riesz representation theorem, for every $x \in [a,b]$ there is a finite non-negative measure $\mu_x$ such that

$$L(f,x)=\int f(t)\,\mu_x(dt), \qquad \text{for all} \quad f \in C^n([a,b]).$$

Naturally, the converse is not true. That is, only special kernels $\mu_x(\cdot)$ will transform continuous functions into continuous functions.

COROLLARY 2.13. *Consider the positive linear operator*

$$L\colon C^n([a,b]) \to C([a,b]), \qquad n \in \mathbb{N}.$$

*Let*

$$\begin{aligned} c_k(x) &= L((t-x)^k, x), \qquad k=0,1,\ldots,n; \\ d_n(x) &= [L(|t-x|^n, x)]^{1/n}; \qquad\qquad (2.13.1)\\ c(x) &= \max(x-a, b-x) \ \ (c(x) \geqslant (b-a)/2). \end{aligned}$$

*Let* $f \in C^n([a,b])$ *such that* $\omega_1(f^{(n)}, h) \leqslant w$, *where* $w, h$ *are fixed positive numbers*, $0<h<b-a$. *Then we have the upper bound*

$$|L(f,x)-f(x)| \leqslant |f(x)|\,|c_0(x)-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(x)|}{k!}\,|c_k(x)| + R_n. \qquad (2.13.2)$$

*Here*

$$R_n = w\phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n = \frac{w}{n!}\,\Theta_n(h/c(x))\,d_n^n(x), \qquad (2.13.3)$$

*where*

$$\Theta_n(h/u) = n!\,\phi_n(u)/u^n.$$

*The above inequality is sharp. Analogous to Theorem* 2.12, *it is in a certain sense attained by* $w\phi_n((t-x)_+)$ *and a measure* $\mu_x$ *supported by* $\{x, b\}$ *when* $x-a \leqslant b-x$, *also attained by* $w\phi_n((x-t)_+)$ *and a measure* $\mu_x$ *supported by*

*$\{x, a\}$ when $x-a \geqslant b-x$: in each case with masses $c_0(x)-(d_n(x)/c(x))^n$ and $(d_n(x)/c(x))^n$, respectively.*

*Proof.* Apply Theorem 2.12 with 0 shifted to $x$. ∎

The next lemma will be used in Theorem 2.15.

LEMMA 2.14. *Be given $[a, b] \subset \mathbb{R}$ and $x_0 \in (a, b)$ fixed, consider all measures $\mu$ with prescribed moments*

$$\mu([a, b]) = c_0 > 0; \quad \int (t-x_0)\,\mu(dt) = c_1(x_0), \quad \int |t-x_0|\,\mu(dt) = d_1(x_0) > 0. \tag{2.14.1}$$

*For $w, h > 0$ $(0 < h < b-a)$ as given numbers, put $M(x_0) = \sup_\mu \int (w/c_0)\,\phi_1(|t-x_0|)\,\mu(dt)$.*

*Then*

$$M(x_0) = w\phi_1(b-x_0)\left(\frac{d_1(x_0)+c_1(x_0)}{2c_0(b-x_0)}\right) + w\phi_1(x_0-\alpha)\left(\frac{d_1(x_0)-c_1(x_0)}{2c_0(x_0-a)}\right). \tag{2.14.2}$$

*The optimal measure is carried by $\{a, x_0, b\}$.*

*Proof.* Easy. ∎

The assertion of Theorem 2.12 can be improved if more is known about $\mu$. One result in this direction is the following.

THEOREM 2.15. *Let $[a, b] \subset \mathbb{R}$, $x_0 \in (a, b)$, and consider all measures $\mu$ on $[a, b]$ such that*

$$\mu([a, b]) = c_0 > 0; \quad \int (t-x_0)\,\mu(dt) = c_1(x_0), \quad \int |t-x_0|\,\mu(dt) = d_1(x_0) > 0. \tag{2.15.1}$$

*Further, consider $f \in C^1([a, b])$ with $w_1(f', h) \leqslant w$ where $w$, $h$ are given positive numbers $(0 < h < b-a)$.*

*Then, we get the best upper bound*

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\,|c_0-1| + |f'(x_0)|\,|c_1(x_0)| + c_0 M(x_0), \tag{2.15.2}$$

*where $M(x_0)$ is given by (2.14.2).*

*Proof.* Easy. ∎

## IV. Nearly Attainable Inequalities

Here we establish some good inequalities with explicit constants better than those in the literature. They involve the first modulus of continuity, or its (smaller) modification, of the $n$th derivative of $f \in C^n([a, b])$, $n \geqslant 1$, evaluated at $h = rd_{n+1}(x_0)$, where $d_{n+1}(x_0) = (\int |t - x_0|^{n+1} \mu(dt))^{1/(n+1)}$, $x_0 \in [a, b]$.

The following is a refinement of a result due to B. Mond and R. Vasudevan [23].

Theorem 2.16. *Consider a closed interval $[a, b] \subset \mathbb{R}$, a given point $x_0 \in [a, b]$, and a measure $\mu$ on $[a, b]$ of mass $m > 0$ satisfying*

$$\int (t - x_0)\, \mu(dt) = 0; \qquad \left(\int (t - x_0)^2\, \mu(dt)\right)^{1/2} = d_2(x_0) > 0.$$

*Then for $f \in C^1([a, b])$ and $r > 0$,*

$$\left|\int f\, d\mu - f(x_0)\right| \leqslant |f(x_0)|\, |m - 1| + \left(\sqrt{m} + \frac{1}{r}\right) \omega_1(f', rd_2(x_0))\, d_2(x_0). \tag{2.16.1}$$

*Proof.* By the mean value theorem, there is $\xi \in (t, x_0)$ such that

$$f(t) - f(x_0) = (t - x_0) f'(x_0) + (t - x_0)(f'(\xi) - f'(x_0)). \tag{2.16.2}$$

Then

$$|f'(\xi) - f'(x_0)| \leqslant \omega_1(f', |\xi - x_0|) \leqslant \omega_1(f', |t - x_0|) = \omega_1(f', |t - x_0|\, \delta^{-1}\delta)$$
$$\leqslant (1 + |t - x_0|\, \delta^{-1})\, \omega_1(f', \delta) \qquad (\text{for all } \delta > 0).$$

Therefore $|f'(\xi) - f'(x_0)| \leqslant (1 + |t - x_0|\, \delta^{-1})\, \omega_1(f', \delta)$, for all $\delta > 0$. Multiplying by $|t - x_0|$, integrating relative to $\mu$, and appling (2.16.2) we have

$$\left|\int f\, d\mu - mf(x_0)\right| \leqslant |f'(x_0)| \left|\int (t - x_0)\, \mu(dt)\right|$$
$$+ \left(\int |t - x_0|\, \mu(dt) + \left(\int (t - x_0)^2\, \mu(dt)\right) \delta^{-1}\right) \omega_1(f', \delta)$$
$$\leqslant |f'(x_0)| \left|\int (t - x_0)\, \mu(dt)\right|$$
$$+ \left[\left(\int (t - x_0)^2\, \mu(dt)\right)^{1/2} \sqrt{m} + \delta^{-1} \int (t - x_0)^2\, \mu(dt)\right]$$
$$\cdot\, \omega_1(f', \delta).$$

Letting $\delta = rd_2(x_0)$, one obtains (2.16.1). ▮

The following is a refinement of the main theorem of a paper due to H. H. Gonska [12].

Gonska's result is the latest improvement in this type of inequality. Here $\mu(f) = \int f\,d\mu$.

THEOREM 2.17. *Let $\mu$ be a measure on $[a, b]$ of mass $m > 0$. Consider $0 < h < b - a$ and $x_0 \in [a, b]$ and let $f \in C^1([a, b])$.*

*Then*

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\,|m-1| + \omega_1(f', h)\left\{\mu(|t-x_0|) + \frac{1}{2h}\,\mu((t-x_0)^2)\right\}$$

$$+\left|\frac{1}{2h}\int_{-h}^{h} f'_*(x_0+u)\,du\right|\,|\mu(t-x_0)|, \qquad (2.17.1)$$

*where*

$$\begin{aligned} f'_*(t) &= f'(a), && t<a; \\ &= f'(t), && t \in [a, b]; \\ &= f'(b), && t > b. \end{aligned}$$

*Remark.* Observe that $|(1/2h)\int_{-h}^{h} f'_*(x_0+u)\,du| \leqslant \|f'\|$, where $\|\cdot\|$ denotes the sup-norm over $[a, b]$.

*Proof.* If $g \in C^2([a, b])$, then clearly

$$\left|\int g\,d\mu - mg(x_0)\right| \leqslant |g'(x_0)|\left|\int (t-x_0)\,\mu(dt)\right| + \frac{\|g''\|}{2}\int (t-x_0)^2\,\mu(dt). \qquad (2.17.2)$$

Further,

$$|\mu(f) - f(x_0)\,m| \leqslant |\mu(f-g) - m(f-g)(x_0)| + |\mu(g) - g(x_0)\,m|.$$

Hence,

$$|\mu(f) - f(x_0)\,m| \leqslant \|(f-g)'\|\,\mu(|t-x_0|)$$

$$+ |g'(x_0)|\,|\mu(t-x_0)| + \frac{\|g''\|}{2}\,\mu((t-x_0)^2). \quad (2.17.3)$$

Let $f'_h$ be the so-called *first Stekloff function* of $f'$; that is,

$$(f'_h)(t) = \frac{1}{2h}\int_{-h}^{h} f'_*(t+u)\,du, \qquad (a \leqslant t \leqslant b).$$

By a well-known theory [29],

$$\|f' - (f')_h\| \leqslant \omega_1(f', h) \qquad \text{and} \qquad \|(f')'_h\| \leqslant h^{-1}\omega_1(f', h).$$

It is easy to find $g \in C^2([a, b])$ such that $g' = (f')_h$. Applying (2.17.3), one obtains

$$|\mu(f) - f(x_0)\,m| \leqslant \omega_1(f', h)\left\{\mu(|t - x_0|) + \frac{1}{2h}\,\mu((t - x_0)^2)\right\}$$
$$+ \left|\frac{1}{2h}\int_{-h}^{h} f'_*(x_0 + u)\,du\right|\,|\mu(t - x_0)|. \quad \blacksquare$$

COROLLARY 2.18. *Let the closed interval* $[a, b] \subset \mathbb{R}$ *and* $x_0 \in [a, b]$. *Also consider a measure* $\mu$ *on* $[a, b]$ *of mass* $m > 0$, *such that*

$$\int (t - x_0)\,\mu(dt) = 0; \qquad \left(\int (t - x_0)^2\,\mu(dt)\right)^{1/2} = d_2(x_0) > 0.$$

*Let* $r > 0$ *and* $f \in C^1([a, b])$. *Then*

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\,|m - 1| + \left(\sqrt{m} + \frac{1}{2r}\right)\omega_1(f', rd_2(x_0))\,d_2(x_0). \tag{2.18.1}$$

*Observe that* (2.18.1) *is sharper that* (2.16.1).

*Proof.* By Schwarz's inequality $\mu(|t - x_0|) \leqslant (\mu((t - x_0)^2))^{1/2}\sqrt{m}$. Now apply (2.17.1) with $h = rd_2(x_0)$. $\blacksquare$

Using (2.3.5), we obtain the following result.

THEOREM 2.19. *Let* $f \in C^1([a, b])$ *and* $\mu$ *be a measure on* $[a, b]$ *of mass* $m > 0$ *with given moments*

$$\int (t - x_0)\,\mu(dt) = 0; \qquad \left(\int (t - x_0)^2\,\mu(dt)\right)^{1/2} = d_2(x_0) > 0,$$

*where* $x_0 \in [a, b]$. *Consider* $r > 0$. *Then*

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\,|m - 1| + \frac{1}{8r}\,(2 + \sqrt{mr})^2\,\omega_1(f', rd_2(x_0))\,d_2(x_0). \tag{2.19.1}$$

*Note.* If $x_0=0$ we could get a sharper inequality by using the modified modulus of continuity $\bar{\omega}_1$ instead of $\omega_1$, where

$$\bar{\omega}_1(f', h)=\sup\{|f'(x)-f'(y)|: x\cdot y\geqslant 0, |x-y|\leqslant h\}. \quad (2.19.2)$$

Obviously, $\bar{\omega}_1\leqslant \omega_1$.

COROLLARY 2.20. *In the special case of $m=1$, $x_0=0$ we have*

$$\left|\int f\,d\mu-f(0)\right|\leqslant \bar{\omega}_1(f', rd_2)\,\frac{(2+r)^2}{8r}\,d_2 \quad (2.20.1)$$

(*where $d_2=d_2(0)$*).

*Proof of Theorem* 2.19. Integrating (2.9.1) relative to $\mu$ we get

$$\int f\,d\mu-mf(x_0)=f'(x_0)\int (t-x_0)\,\mu(dt)+\int K_1(t, x_0)\,\mu(dt), \quad (2.19.3)$$

where

$$K_1(t, x_0)=\int_{x_0}^{t}\phi(x)\,dx; \qquad \phi(x)=f'(x)-f'(x_0).$$

Note that $\phi(x_0)=0$ and $|\phi(x)|\leqslant \omega_1(f', h)\lceil |x-x_0|/h\rceil$ for all $h>0$.
Hence,

$$|K_1(t, x_0)|\leqslant M_1(t, x_0)=\omega_1(f', h)\int_{x_0}^{t}\left\lceil \frac{x-x_0}{h}\right\rceil dx,$$

for all $t, x_0\in[a, b]$. By (2.3.5), we obtain

$$|K_1(t, x_0)|\leqslant M_1(t, x_0)\leqslant \omega_1(f', h)\left[\frac{(t-x_0)^2}{2h}+\frac{|t-x_0|}{2}+\frac{h}{8}\right]. \quad (2.19.4)$$

Now integrating (2.19.4) against $\mu$, using Schwarz's inequality, and setting $h=rd_2(x_0)$, we find

$$\int |K_1(t, x_0)|\,\mu(dt)\leqslant \frac{1}{8r}(2+\sqrt{mr})^2\,\omega_1(f', rd_2(x_0))\,d_2(x_0). \quad (2.19.5)$$

Finally, from (2.19.5) and (2.19.3)

$$\left|\int f\,d\mu-mf(x_0)\right|\leqslant \frac{1}{8r}(2+\sqrt{mr})^2\,\omega_1(f', rd_2(x_0))\,d_2(x_0).$$

Since

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\;|m-1| + \left|\int f\,d\mu - f(x_0)\,m\right|$$

the theorem follows. ▮

*Remark* 2.21. When $\omega_1(f', h) = Ah^\alpha$, $0 < \alpha \leqslant 1$ and $A > 0$ constant, the value of $r > 0$ minimizing the right-hand side of (2.19.1) is given by

$$r = 2(1-\alpha)/(\sqrt{m}(1+\alpha)). \tag{2.21.1}$$

If $\alpha = 1$ then letting $r \downarrow 0$ one obtains that

$$\left|\int f\,d\mu - f(x_0)\right| \leqslant |f(x_0)|\;|m-1| + \frac{1}{2}\,Ad_2^2(x_0), \tag{2.21.2}$$

where

$$A = \sup_{s \neq t}\{|f'(s) - f'(t)|/|s-t|\}.$$

When $|a| = b$ and $x_0 = 0$, the last inequality is attained by $f(x) = x^2$ and a measure $\mu$ with masses $m/2$ at $\pm b$ (both sides are then equal $mb^2$).

*Remark* 2.22. When $m = 1$, $|a| = b$, $x_0 = 0$, $r = 2$, and $\alpha$ is small, the inequality (2.19.1) is nearly attained by $f(x) = |x|^{1+\alpha}$ and $\mu$ with mass $\frac{1}{2}$ at $\pm b$.

*Remark* 2.23. With $m = 1$, $|a| = b$, $x_0 = 0$, the measure $\mu$ having mass $\frac{1}{2}$ at $\pm b$, and $f(x) = |x|^{1+\alpha}$ $(0 < \alpha \leqslant 1)$, the left-hand side of (2.19.1) equals $a^{1+\alpha}$, while the right-hand side equals $(1+\alpha)(r^{\alpha-1}/8)(2+r)^2\,a^{1+\alpha}$. Minimizing over $r$ the right-hand side becomes $C(\alpha)\,a^{1+\alpha}$, where $C(\alpha) = 2^\alpha(1-\alpha)^{-1+\alpha}(1+\alpha)^{-\alpha}$. The quantity $\ln C(\alpha)$ is a concave function of $\alpha$ taking its largest value at $\alpha = 0.580332$ and there $C(\alpha) = 1.650485$. Further

| | |
|---|---|
| $C(0.01) = 1.016923$ | $C(0.6) = 1.649385$ |
| $C(0.05) = 1.084313$ | $C(0.7) = 1.607942$ |
| $C(0.1) = 1.167200$ | $C(0.8) = 1.501667$ |
| $C(0.2) = 1.324023$ | $C(0.9) = 1.318405$ |
| $C(0.3) = 1.46069$ | $C(0.95) = 1.189863$ |
| $C(0.4) = 1.56700$ | $C(0.99) = 1.052338$ |
| $C(0.5) = 1.63299$ | $C(0.999) = 1.0074349.$ |

So we see that (2.19.1) is never far off, in that it is attained up to a factor 1.65 at most.

*Note.* (i) If $0<r\leqslant 4/\sqrt{m}$, then inequality (2.19.1) is sharper than inequality (2.18.1). (ii) If $r\geqslant 4/\sqrt{m}$, then (2.18.1) is sharper than (2.19.1). Because of (2.21.1) case (i) is probably more interesting.

In terms of best constants, in this type of inequality, the next result improves all the related results we are aware of.

COROLLARY 2.24. *Let $x_0\in[a,b]$ and the measure $\mu$ on $[a,b]$ of mass $m>0$ satisfy the moment conditions*

$$\int (t-x_0)\,\mu(dt)=0 \qquad \text{and} \qquad d_2(x_0)=\left(\int (t-x_0)^2\,\mu(dt)\right)^{1/2}.$$

*Consider $r>0$ and $f\in C^1([a,b])$. Then*

$$\begin{aligned}\left|\left|\int f\,d\mu-f(x_0)\right|-|f(x_0)|\,|m-1|\right.&\\ &\leqslant \frac{1}{8r}\,(2+\sqrt{m}r)^2\,\omega_1(f',rd_2(x_0))\,d_2(x_0), \qquad \text{if} \quad r\leqslant 2/\sqrt{m};\\ &\leqslant \sqrt{m}\omega_1(f',rd_2(x_0))\,d_2(x_0), \qquad \text{if} \quad r>2/\sqrt{m}. \qquad (2.24.1)\end{aligned}$$

*When $x_0=0$, we get a sharper estimate by replacing $\omega_1$ by $\bar{\omega}_1$ (see (2.19.2)).*

*Proof.* Note that the first part of (2.24.1) follows from (2.19.1). If $r\geqslant 2/\sqrt{m}$ then apply (2.19.1) with $r$ replaced by $r_1=2/\sqrt{m}$ and note that $(1/8r_1)(2+\sqrt{m}r_1)^2=\sqrt{m}$. ∎

Taking $r=\frac{1}{2}$ in (2.19.1) one obtains:

THEOREM 2.25. *Let the random variable $Y$ have distribution $\mu$, $E(Y)=x_0$, and $\mathrm{Var}(Y)=\sigma^2$. Consider $f\in C^1_B(\mathbb{R})$. Then*

$$|Ef(Y)-f(x_0)|=\left|\int f\,d\mu-f(x_0)\right|\leqslant (1.5625)\,\omega_1\left(f',\frac{1}{2}\sigma\right)\sigma. \qquad (2.25.1)$$

*The last inequality is stronger than the corresponding pointwise results following B. Mond and R. Vasudevan [23] and J. P. King [17].*

2.26. *Application*

Consider $X_j$ real i.i.d. random variables and put $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$. Let $x_0 = E(X)$, $\sigma^2 = \mathrm{Var}(X)$ thus $E(S_n/n) = x_0$ and $\mathrm{Var}(S_n/n) = \sigma^2/n$. Denote $F_{S_n}$ the d.f. of $S_n$. Then (2.25.1) yields

$$|Ef(S_n/n) - f(x_0)|$$

$$= \left| \int_{-\infty}^{\infty} f(t/n)\, dF_{S_n}(t) - f(x_0) \right| \leqslant (1.5625)\, \omega_1 \left( f', \frac{\sigma}{2\sqrt{n}} \right) \frac{\sigma}{\sqrt{n}}. \tag{2.26.1}$$

The following Corollaries 2.27, 2.28, 2.29, and 2.30 are applications of (2.26.1) to well-known positive linear operators arising from probability theory. The corollaries about the Baskakov, Szász-Mirakjan, and Weierstrass operators are improvements of the corresponding results from Z. Ditzian [8] and S. P. Singh [28].

We start with the classical Bernstein polynomials.

COROLLARY 2.27. *For any* $f \in C^1([0, 1])$ *consider* $(B_n f)(t) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} t^k (1-t)^{n-k}$, $t \in [0, 1]$. *Then*

$$|(B_n f)(t) - f(t)| \leqslant (1.5625)\, \omega_1 \left( f', \frac{1}{2} \sqrt{\frac{t(1-t)}{n}} \right) \sqrt{\frac{t(1-t)}{n}}$$

$$\leqslant \left( \frac{0.78125}{\sqrt{n}} \right) \omega_1 \left( f', \frac{1}{4\sqrt{n}} \right).$$

*Proof.* Consider $(X_j)_{j \in \mathbb{N}}$ Bernoulli (i.i.d) random variables such that $Pr\ (X_j = 0)\ = 1 - t$, $Pr\ (X_j = 1)\ = t$, $t \in (0, 1)$, then $E(X) = t$ and $\mathrm{Var}(X) = t(1-t)$. Now apply (2.26.1) with $x_0 = t$. Further, note that $\max_{0 \leqslant t \leqslant 1} (t(1-t)) = \frac{1}{4}$ at $t = \frac{1}{2}$. ∎

For $t \geqslant 0$ and $f \in C_B(\mathbb{R}^+)$ the Szász–Mirakjan operator is defined as

$$(M_n f)(t) = e^{-nt} \sum_{k=0}^{\infty} f\left( \frac{k}{n} \right) \frac{(nt)^k}{k!}$$

while the Baskakov-type operator is defined as

$$(V_n f)(t) = \sum_{k=0}^{\infty} f\left( \frac{k}{n} \right) \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}.$$

Both operators are of the form $E(S_n/n)$ above. Namely, $X$ there has the distribution

$$P_X = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_k \qquad \text{and} \qquad P_X = \sum_{k=0}^{\infty} \left(\frac{1}{1+t}\right)\left(\frac{t}{1+t}\right)^k \delta_k,$$

(Poisson and geometric), respectively. In both cases, $E(X) = t$ while $\operatorname{Var}(X) = t$ and $\operatorname{Var}(X) = (t + t^2)$, respectively.

Thus (2.26.1) implies:

COROLLARY 2.28. *With the above notations, we have*

$$|(M_n f)(t) - f(t)| \leqslant (1.5625)\, \omega_1\left(f', \frac{1}{2}\left(\frac{t}{n}\right)^{1/2}\right)\left(\frac{t}{n}\right)^{1/2}$$

*and*

$$|(V_n f)(t) - f(t)| \leqslant (1.5625)\, \omega_1\left(f', \frac{1}{2}\left(\frac{t+t^2}{n}\right)^{1/2}\right)\left(\frac{t+t^2}{n}\right)^{1/2},$$

*for all* $f \in C_B^1(\mathbb{R}^+)$.

*The Weierstrass operator is defined by*

$$(W_n f)(t) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-n(x-t)^2}\, dx.$$

*It agrees with* $Ef(S_n/n)$ *when* $X$ *has the normal distribution* $(t, \frac{1}{2})$ *with density* $(1/\sqrt{\pi})\, e^{-(x-t)^2}$.

COROLLARY 2.29. *For all* $f \in C_B^1(\mathbb{R})$ *we have*

$$\| W_n(f) - f \| \leqslant (1.5625)\, \omega_1\left(f', \frac{1}{2\sqrt{2n}}\right)\frac{1}{\sqrt{2n}},$$

*where* $\|\cdot\|$ *is the* sup-*norm.*

*As our last illustration, let* $X$ *have an exponential density* $e^{-x/t}$ *on* $\mathbb{R}^+$ *so that* $E(X) = t$, $\operatorname{Var}(X) = t^2$. *Then* $S_n$ *has a gamma density with parameters* $n$ *and* $t^{-1}$, *so that* $S_n/n$ *has a gamma density with parameters* $n$ *and* $n/t$.

*This leads to the operator (see* [11, *p.* 219])

$$(H_n f)(t) = \frac{n^n}{(n-1)!\, t^n} \int_0^{\infty} f(x)\, x^{n-1} e^{-nx/t}\, dx, \qquad t > 0.$$

COROLLARY 2.30. *For $f \in C_B^1(\mathbb{R}^+)$, $t > 0$ we find*

$$|(H_n f)(t) - f(t)| \leqslant (1.5625)\, \omega_1\left(f', \frac{t}{2\sqrt{n}}\right) \frac{t}{\sqrt{n}}.$$

In the following we establish similar inequalities for higher derivatives ($n \geqslant 1$).

THEOREM 2.31. *Let $\mu$ be a positive measure of mass $m > 0$ on the closed interval $[a, b] \subset \mathbb{R}$, for which we asume that $((1/m) \int |t - x_0|^{n+1} \mu(dt))^{1/(n+1)} = d_{n+1} > 0$, where $x_0 \in [a, b]$ is fixed. Consider $f \in C^n([a, b])$, $n \geqslant 1$, with $\omega_1(f^{(n)}, rd_{n+1}) \leqslant w$, where $r$, $w$ are given positive numbers. Then*

$$\left|\int f\, d\mu - f(x_0)\right| \leqslant |m - 1|\, |f(x_0)| + \sum_{k=1}^{n} \frac{|f^{(k)}(x_0)|}{k!} \left|\int (t - x_0)^k \mu(dt)\right|$$
$$+ \frac{mw}{rn!}\left[\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{(n+1)}\right] d_{n+1}^n. \tag{2.31.1}$$

*Note.* (i) When $x_0 = 0 \in [a, b]$ then (2.31.1) is also true when $\omega_1$ is replaced by $\bar{\omega}_1$.

(ii) In applications $r$ is usually small.

(iii) Inequality (2.31.1) on $[-b, b]$ for $m = 1$, $r \downarrow 0$, and $x_0 = 0$ is attained by $f(x) = |x|^{n+1}$ and $\mu$ with mass $\frac{1}{2}$ at $\pm b$.

*Proof.* Exactly as the proof of Theorem 2.10 except that we use the bound (2.3.6) for $\phi_n$ and take $h = rd_{n+1}$ instead. ∎

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